THE BEST CONSTANT FOR THE CENTERED MAXIMAL OPERATOR ON RADIAL DECREASING FUNCTIONS

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Abstract. We show that the lowest constant appearing in the weak type (1,1) inequality satisfied by the centered Hardy-Littlewood maximal operator on radial, radially decreasing integrable functions is 1.

1. Introduction

A considerable amount of work has been devoted in the literature to finding good bounds, or best bounds if possible, in the inequalities satisfied by the several variants of the Hardy-Littlewood maximal operator. We mention, for instance, [A1], [A2], [A3], [A4], [A5], [ACP], [AlPe], [AlPe2], [AV], [Bou1], [Bou2], [Bou3], [Ca], [CF], [CLM], [GK], [GM], [GMM], [Ki], [Me1], [Me2], [Mu], [St1], [St2], [St3], [StSt]. Additional references can be found in the aforementioned papers.

Let M_d be the centered maximal operator (cf. (1) below for the definition) associated to euclidean balls and Lebesgue measure on $\mathbb{R}^d$. It is well known that if $1 < p \le \infty$, then there exists a constant $c_{p,d}$ such that for all $f \in L^p(\mathbb{R}^d)$, $\|M_d f\|_p \le c_{p,d}\|f\|_p$, and the problem lies in determining the lowest such $c_{p,d}$. When $p = \infty$, trivially we can take $c_{p,d} = 1$ for all d, since averages never exceed a supremum, while if $p = 1$, then $M_d f \notin L^1(\mathbb{R}^d)$ unless $f = 0$ almost everywhere. So for $p = 1$ one considers instead the best constant c_d appearing in the weak type $(1,1)$ inequality (cf. 3 below). Obviously, $c_d \ge 1$, since by the Lebesgue Differentiation Theorem, $M_d f \ge |f|$ a.e. whenever $f \in L^1(\mathbb{R}^d)$. We shall see that if we impose on f the additional conditions of being radial and radially decreasing, then actually $c_d = 1$ for every dimension d. This improves on the previously known upper bound $c_d \le 4$ (which nevertheless holds for all radial functions, not necessarily decreasing, cf. [MS, Theorem 3]).

Our result is obtained by identifying the extremal case: For the class of radial, radially decreasing functions f of norm one, the Dirac delta "function" δ is extremal. That is, $M_d\delta(x) \ge M_d f(x)$ for every x. Since $M_d\delta$ can be easily computed, and it yields a best constant equal to 1, the result follows. Without the decreasing assumption on f, the value of the best constant is not known (as indicated above, it is bounded by 4).

Regarding the dependency of $c_{p,d}$ on d, for general functions, E. M. Stein showed that when $1 < p < \infty$, the constants $c_{p,d} < \infty$ could be chosen to be uniform in d ([St1], [St2], see also [St3]).

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With respect to weak type (1,1) bounds, best constants grow at most like $O(d)$ (cf. [StSt]). Additionally, for the maximal function associated to *cubes* (rather than euclidean balls) it is known that best bounds approach infinity with the dimension (cf. [A5], and also [Au], where it is shown that bounds increase at least as $O(\log^{1-\varepsilon} d)$, for arbitrary $\varepsilon > 0$). While the corresponding problem for euclidean balls has not yet been solved, it seems very likely that uniform bounds do not exist in this case either. Hence the renewed interest in finding natural subspaces of $L^1(\mathbb{R}^d)$ for which bounds independent of d can be obtained, and when uniform bounds are known, in determining their optimal values.

2. Notation and results

Let λ^d denote the Lebesgue measure on $\mathbb{R}^d$, and let $B(x,r)$ be the euclidean *closed* ball centered at x of radius $r > 0$. Thus, $B(x,r)$ is defined using the ℓ_2 distance $\|x\|_2 := \sqrt{x_1^2 + \cdots + x_d^2}$. The centered maximal function $M_d f$ of a locally integrable function f is

$$M_d f(x) := \sup_{r>0} \frac{1}{\lambda^d(B(x,r))} \int_{B(x,r)} |f| d\lambda^d \tag{1}$$

(the choice of closed balls in the definition is mere convenience; using open balls instead does not change the value of $M_d f(x)$). Likewise, the centered maximal function $M_d \mu$ of a locally finite measure μ is

$$M_d \mu(x) := \sup_{r>0} \frac{\mu(B(x,r))}{\lambda^d(B(x,r))}. \tag{2}$$

It is well known that the maximal function satisfies the following weak type $(1,1)$ inequality:

$$\lambda^d(\{M_d f \geq \alpha\}) \leq \frac{c_d \|f\|_1}{\alpha}, \tag{3}$$

where c_d does not depend on $f \in L^1(\mathbb{R}^d)$.

We denote the average of the function h over the set E by

$$h_E := \frac{1}{\lambda^d(E)} \int_E h d\lambda^d. \tag{4}$$

Likewise, the average (with respect to Lebesgue measure) of the measure μ over the set E is denoted by

$$\mu_E := \frac{\mu(E)}{\lambda^d(E)}. \tag{5}$$

The next "geometric lemma on averages", states the intuitively plausible fact that for a radial decreasing function on $\mathbb{R}^d$, the average over any ball B centered at zero is at least as large as the average over any other ball with center outside B (or on its border). By decreasing we mean non-strictly decreasing.

Lemma 2.1. *Let $f : (0,\infty) \to (0,\infty)$ be a decreasing function. Define $g : \mathbb{R}^d \to \mathbb{R}$ by setting $g(x) := f(\|x\|_2)$. If g is locally integrable, then for every pair of radii $R, r > 0$, and every $y \in \mathbb{R}^d$ with $\|y\|_2 \ge R$, we have*

$$g_{B(0,R)} \ge g_{B(y,r)}. \tag{6}$$

Remark 2.2. Actually, for the application below we only need the case $r < R$, but since the result is also true when $R \le r$, we do not exclude this from the statement of the lemma.

Remark 2.3. Obviously, if f is locally integrable then so is g. Local integrability of g is all we need, so we only assume this weaker condition.

Remark 2.4. It is natural to ask whether the hypothesis that y does not belong to the interior of $B(0,R)$ can be relaxed to $B(y,r) \setminus B(0,R) \ne \emptyset$. In fact, it is easy to see that the latter condition is not enough, even in one dimension: Let $\psi(x) := (1 - |x|)_+$ be the positive part of $1 - |x|$. Then $\psi_{[-1,1]} < \psi_{[-1/2,1]}$, so if $\varepsilon > 0$ is sufficiently small, we also have $\psi_{[-1,1]} < \psi_{[-1/2,1+\varepsilon]}$.

Remark 2.5. In order to obtain large averages, one must integrate over the parts of the space where a function is large. And this is so no matter what measure is used. Thus, it is tempting to conjecture that Lemma 2.1 actually holds for a large class of measures, rather than just Lebesgue's. While this may be the case, some condition on the measure is needed, as the following example shows.

Let $d = 2$ and set $\mu(A) := \lambda^2(A \cap B(0,1))$, i.e., μ is the restriction of planar Lebesgue measure to the unit ball. Let $\psi(x) := (1 - \|x\|_2)_+$, and observe that $\psi_{B(0,1)} < \psi_{B(e_1,1)}$. This is so since $\psi_{B(0,1)}$ is exactly equal to the average over the cone C contained in $B(e_1,1) \cap B(0,1)$ and between the lines $y = \pm\sqrt{3}x$, while obviously $\psi_C < \psi_{B(e_1,1)\cap B(0,1)}$. But $\mu(B(e_1,1) \cap B(0,1)) = \mu(B(e_1,1))$, so $\psi_{B(e_1,1)\cap B(0,1)} = \psi_{B(e_1,1)}$. Therefore, Lemma 2.1 does not extend to all radial measures.

Remark 2.6. Another natural attempt to generalize Lemma 2.1 is to consider norms different from the euclidean one. Since the most often used maximal functions on $\mathbb{R}^d$ are defined either using euclidean balls or cubes with sides parallel to the axes, i.e., ℓ_∞ balls, the case of the ℓ_∞ norm $\|x\|_\infty := \max\{|x_1|, \ldots, |x_d|\}$ is particularly interesting. Here a function $g : \mathbb{R}^d \to \mathbb{R}$ is radial decreasing if there exists a decreasing function $f : (0,\infty) \to (0,\infty)$ such that $g(x) = f(\|x\|_\infty)$. Likewise, the corresponding maximal operator is obtained by averaging over ℓ_∞ balls B_∞ in (1), instead of using ℓ_2 balls.

The following example, due to Professor Guillaume Aubrun and included here with his permission, shows that Lemma 2.1 fails when $\|\cdot\|_2$ is replaced by $\|\cdot\|_\infty$. Let $g = \chi_{[-2^{-1},2^{-1}]^d}$, let $B_\infty(0,R) = [-4^{-1}3, 4^{-1}3]^d$, and let $B_\infty(y,r) = [4^{-1}, 4^{-1}5] \times [-2^{-1}, 2^{-1}]^{d-1}$, i.e., $y = 3e_1/4$ and $r = 1/2$. Since $2^{d+2} < 3^d$ provided $d \ge 4$, for every $d = 4, 5, \ldots$ we have

$$g_{B_\infty(0,R)} = \frac{2^d}{3^d} < \frac{1}{4} = g_{B_\infty(y,r)}.$$

The preceding inequality is strict, so sufficiently small perturbations of the sets involved will preserve it. Thus, Lemma 2.1 also fails when $\|\cdot\|_2$ is replaced by $\|\cdot\|_p$, provided that p is high enough (perhaps depending on d). Here $\|x\|_p := (|x_1|^p + \cdots + |x_d|^p)^{1/p}$.

Proof of Lemma 2.1. Observe first that the result for $\|y\|_2 \geq R$ can be immediately derived from the special case $\|y\|_2 = R$. To see why, assume it holds for $\|y\|_2 = R$, and suppose $\|w\|_2 > R$. Then

$$g_{B(w,r)} \leq g_{B(0,\|w\|_2)} \leq g_{B(0,R)},$$

since the average over a ball centered at 0 of a radial decreasing function does not decrease when we reduce the radius. So we assume that $\|y\|_2 = R$. Using a change of variables if necessary, we suppose that $R = 1$ (just to simplify expressions). Then we take $y = e_1$, by symmetry; finally, we suppose that f is left continuous. This last assumption is made purely for notational convenience: It entails that nonempty level sets $\{g \geq m\}$ are closed balls, agreeing with our notation $B(0, t)$.

We show that $r^d \int_{B(0,1)} g \geq \int_{B(e_1,r)} g$. To this end, it is enough to prove that for every $m > 0$ the corresponding level sets satisfy

$$r^d \lambda^d(B(0,1) \cap \{g \geq m\}) \geq \lambda^d(B(e_1, r) \cap \{g \geq m\}). \tag{7}$$

If either $m < g(e_1)$, or $m > g(x)$ for all $x \neq 0$, then inequality (7) holds trivially. If $g(e_1) \leq m \leq g(x)$ for some $x \neq 0$, then there exists a $t \in (0, 1]$ such that $\{g \geq m\} = B(0, t)$, so it suffices to show that

$$r^d \lambda^d(B(0, t)) \geq \lambda^d(B(e_1, r) \cap B(0, t)). \tag{8}$$

We assume that $r < 1$ (for otherwise (8) is obvious) and also that $t + r > 1$ (for otherwise $B(e_1, r) \cap B(0, t)$ is the either the empty set or just one point). With these assumptions, the boundaries of the balls $B(e_1, r)$ and $B(0, t)$ are $d - 1$ spheres whose intersection is a $d - 2$ sphere S, with center ce_1 for some $c \in (0, 1)$, and radius ρ. Since $B(e_1, r) \cap B(0, t) \subset B(ce_1, \rho)$, all we need to do is to prove that $\rho \leq rt$, from which (8) follows.

Let us write $x = (x_1, \ldots, x_d)$. Using symmetry, the center and the radius of the sphere S can be determined by considering the intersection of S with the x_1x_2-plane, that is, by simultaneously solving $x_1^2 + x_2^2 = t^2$ and $(x_1 - 1)^2 + x_2^2 = r^2$. Solving for x_1 yields $c = (1 + t^2 - r^2)/2$, and solving for x_2^2, together with some elementary algebraic manipulations, gives

$$\rho^2 = t^2 - \frac{(1 + t^2 - r^2)^2}{4} = (rt)^2 - \frac{(1 - t^2 - r^2)^2}{4} \leq (rt)^2. \tag{9}$$

□

Theorem 2.7. *Let $g \in L^1(\mathbb{R}^d)$ be a radial decreasing function. Then for every $\alpha > 0$,*

$$\lambda^d(\{M_d g > \alpha\}) \leq \frac{\|g\|_1}{\alpha}. \tag{10}$$

Proof. Suppose $\|g\|_1 \neq 0$; using the 1-homogeneity of the maximal operator M_d we see that $\{M_d g > \alpha\} = \{M_d(g/\|g\|_1) > \alpha/\|g\|_1\}$, so we can always replace g with $g/\|g\|_1$. Thus, we assume from the start that $\|g\|_1 = 1$. Let δ denote the Dirac delta mass placed at the origin, i.e., δ is the probability measure defined by $\delta(\{0\}) = 1$. In this case it is easy to compute $M_d\delta$ explicitly: $M_d\delta(x) = 1/\lambda^d(B(x, \|x\|_2))$. Hence, for every $\alpha > 0$ the set $\{M_d\delta \geq \alpha\}$ is a ball, and

$$\lambda^d(\{M_d\delta \geq \alpha\}) = \frac{1}{\alpha}. \tag{11}$$

Inequality (10) is implied by (11), since δ is extremal in the following sense: For every $x \in \mathbb{R}^d$ we have $M_d g(x) \leq M_d\delta(x)$. To see why, note that if $\|x\|_2 = R > 0$ and $r > 0$ is any radius, by Lemma 2.1 we have

$$g_{B(x,r)} \leq g_{B(0,R)} \leq 1/\lambda^d(B(0,R)) = 1/\lambda^d(B(x,R)) = \delta_{B(x,R)}$$

(of course, if $r \geq R$ we do not need the Lemma, since then $0 \in B(x,r)$ and therefore $g_{B(x,r)} \leq \delta_{B(x,r)} \leq \delta_{B(x,R)}$). By taking the supremum over $r > 0$ we conclude that $M_d g(x) \leq M_d\delta(x)$, as was to be shown. □

Using the preceding bound we obtain refined estimates for the operator norm of M_d, from the space of radial decreasing functions in $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$, for $1 < p < \infty$. The proof, a standard Marcinkiewicz interpolation type argument, is omitted (cf., for instance [St3, p. 14]).

Corollary 2.8. *Let $p > 1$ and let $g \in L^p(\mathbb{R}^d)$ be a radial decreasing function. Then*

$$\|M_d g\|_p \leq 2^{\frac{p-1}{p}} \left(\frac{p}{p-1}\right)^{1/p} \|g\|_p. \tag{12}$$

Remark 2.9. Denote by $c_{p,d}$ the operator norm of M_d, from the radial decreasing functions in $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$. Here d is fixed. The preceding result entails that $c_{p,d} = O\left(\frac{p}{p-1}\right)$ as $p \downarrow 1$. Next we show by example that actually $c_{p,d} = \Theta\left(\frac{p}{p-1}\right)$, where Θ denotes the exact order as $p \downarrow 1$. Let f be the characteristic function of the unit ball. Then $\|f\|_p = (\lambda^d(B(0,1)))^{1/p}$. On $B(0,1)$ the maximal function is identically one, while off $B(0,1)$, writing $r = \|x\|_2$, we have $M_d f(x) \geq (r+1)^{-d} \geq (2r)^{-d}$, where the first inequality is obtained by averaging over the smallest ball centered at x that fully contains $B(0,1)$, and the second inequality is used to trivialize integration in polar coordinates. Thus,

$$\int (M_d f)^p \geq |B(0,1)| \left(1 + \frac{1}{(p-1)2^{dp}}\right), \tag{13}$$

so

$$c_{p,d} \geq \left(1 + \frac{1}{(p-1)2^{dp}}\right)^{1/p}. \tag{14}$$

Since $\left(1+\frac{1}{(p-1)2^{dp}}\right)^{1/p}\left(\frac{p-1}{p}\right) \to 2^{-d}$ as $p \downarrow 1$, the assertion about the exact order of $c_{p,d}$ follows.

Remark 2.10. If Lebesgue measure in dimension d is replaced by the standard gaussian measure, or more generally, by any finite measure defined by a bounded, radial, radially decreasing density, the situation is very different: The same example (one delta placed at the origin) shows that constants for the weak type $(1,1)$ inequality grow exponentially fast with the dimension (cf. [A4]) rather than being uniformly bounded by 1. In fact, exponential growth can be shown to hold for some (sufficiently small) values of $p > 1$, simply by using, instead of δ_0, the characteristic function of a small ball centered at 0, and then arguing as in [A4]; a step in this direction is carried, for weak type (p,p) inequalities, in [AlPe3].

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